# Pressure Vessel and Combined Loading

MAE 314 – Solid Mechanics

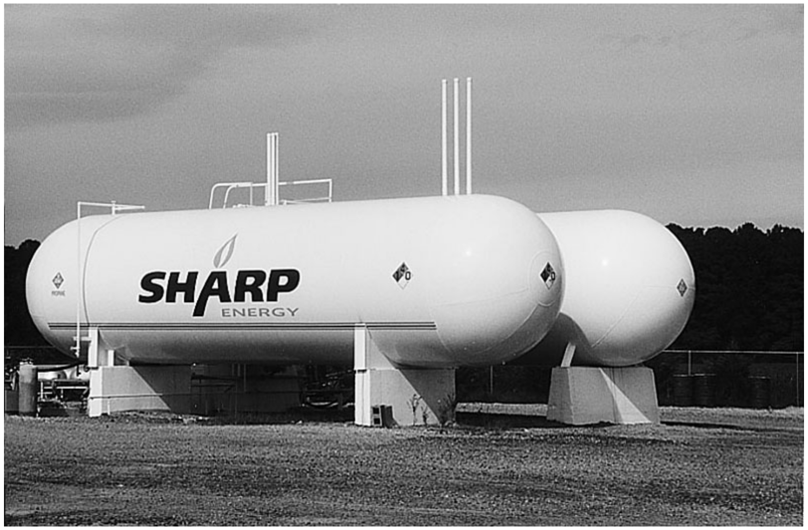
Y. Zhu



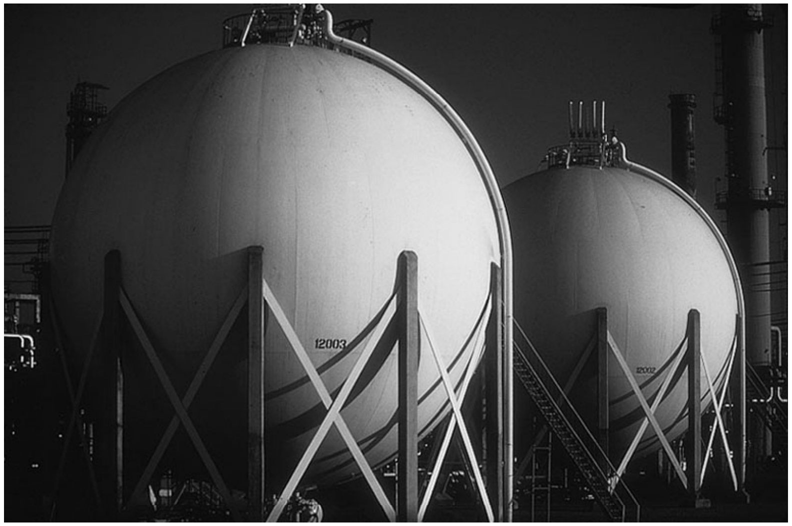
NC STATE achieve! Innovation in Action

NC STATE UNIVERSITY

# Thin-Walled Pressure Vessels




Cylindrical vessel with capped ends



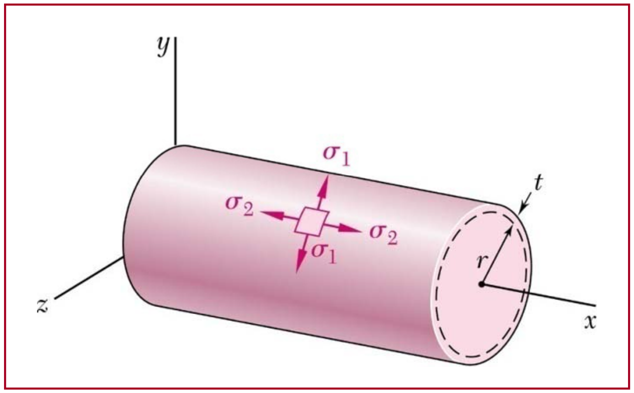

Spherical vessel

- Assumptions
  - Constant gage pressure, p = internal pressure – external pressure
  - Thickness much less than radius (t << r, t / r < 0.1)
  - Internal radius = r
  - Point of calculation far away from ends (St. Venant's principle)



NC STATE achieve! Innovation in Action

# Cylindrical Pressure Vessel

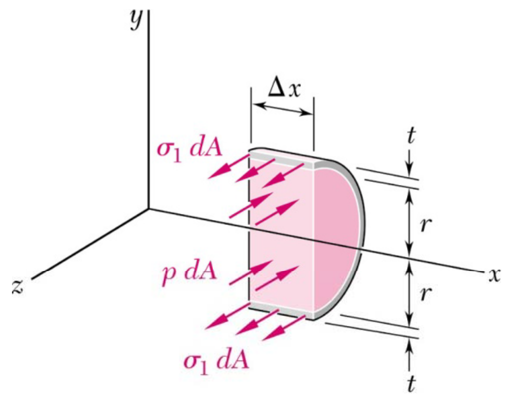



**Circumferential (Hoop) Stress: $\sigma_1$**

Sum forces in the vertical direction.

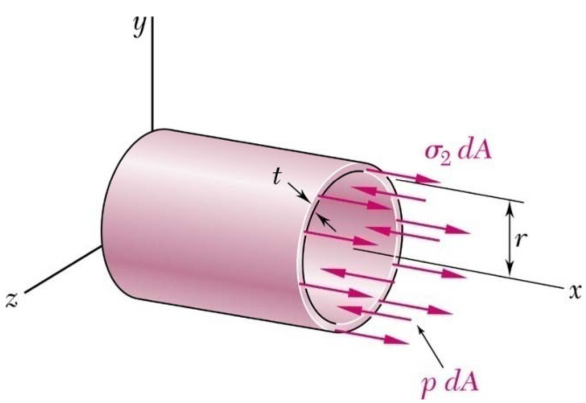
$$2\sigma_1(t\Delta x) - p(2r\Delta x) = 0$$

$$\boxed{\sigma_1 = \frac{pr}{t}}$$




**Longitudinal stress: $\sigma_2$**

Sum forces in the horizontal direction:




$$\sigma_2(2\pi r t) - p(\pi r^2) = 0$$

$$\boxed{\sigma_2 = \frac{pr}{2t}}$$



NC STATE achieve! Innovation in Action

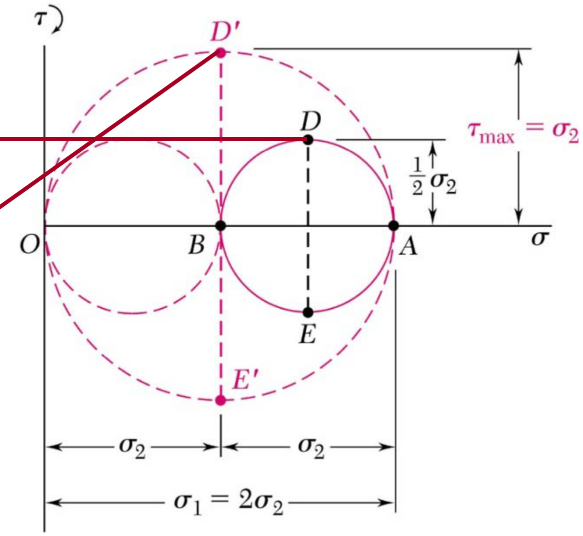
# Cylindrical Pressure Vessel cont'd

- There is also a radial component of stress because the gage pressure must be balanced by a stress perpendicular to the surface.
  - $\sigma_r = p$
  - However $\sigma_r << \sigma_1$ and $\sigma_2$ , so we assume that $\sigma_r = 0$ and consider this a case of plane stress.
- Mohr's circle for a cylindrical pressure vessel:
  - Maximum shear stress (in-plane)

$$\tau_{max} = \frac{\sigma_2}{2} = \frac{pr}{4t}$$

  - Maximum shear stress (out-of-plane)

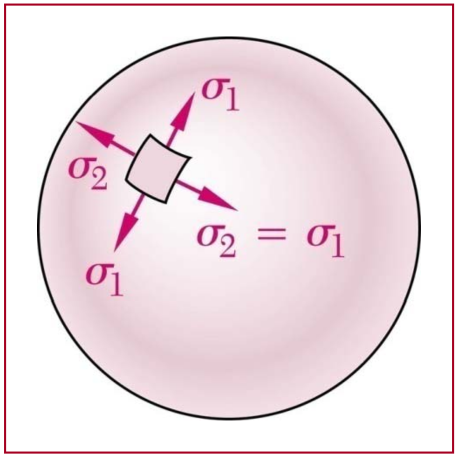
$$\tau_{max} = \sigma_2 = \frac{pr}{2t}$$

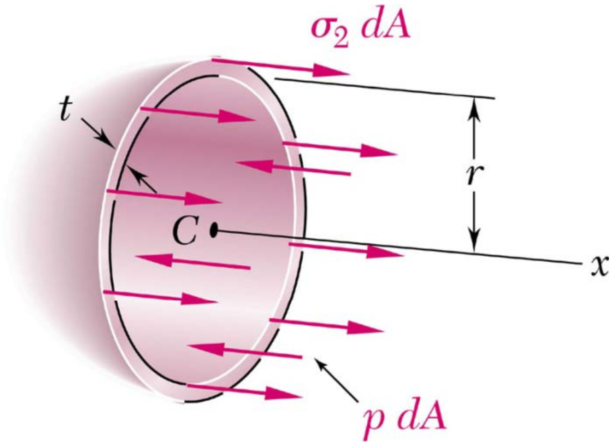


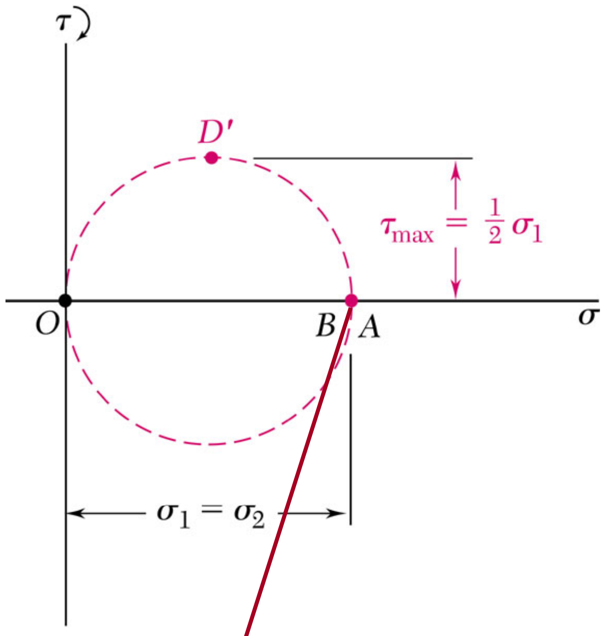



NC STATE achieve! Innovation in Action

# Spherical Pressure Vessel




Sum forces in the horizontal direction:

$$\sigma_2(t2\pi r) - p(\pi r^2) = 0$$

$$\boxed{\sigma_1 = \sigma_2 = \frac{pr}{2t}}$$

In-plane Mohr's circle is just a point

$$\boxed{\tau_{\max} = \frac{\sigma_1}{2} = \frac{pr}{4t}}$$

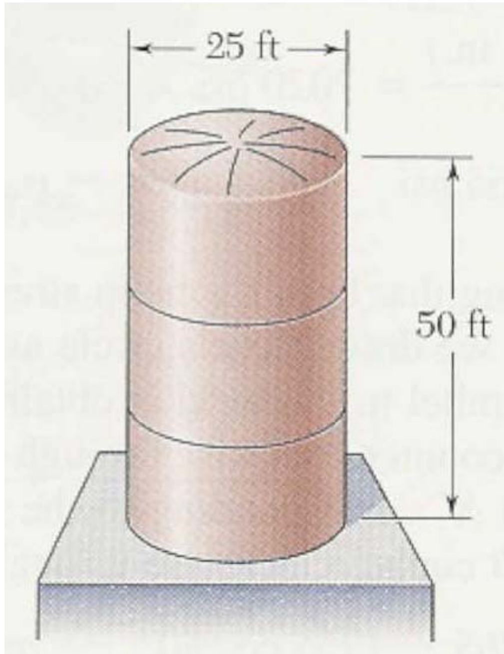


NC STATE achieve! Innovation in Action

NC STATE UNIVERSITY

# Example Problem 1

When filled to capacity, the unpressurized storage tank shown contains water to a height of 48 ft above its base. Knowing that the lower portion of the tank has a wall thickness of 0.625 in, determine the maximum normal stress and the maximum shearing stress in the tank. (Specific weight of water = 62.4 lb/ft$^3$)





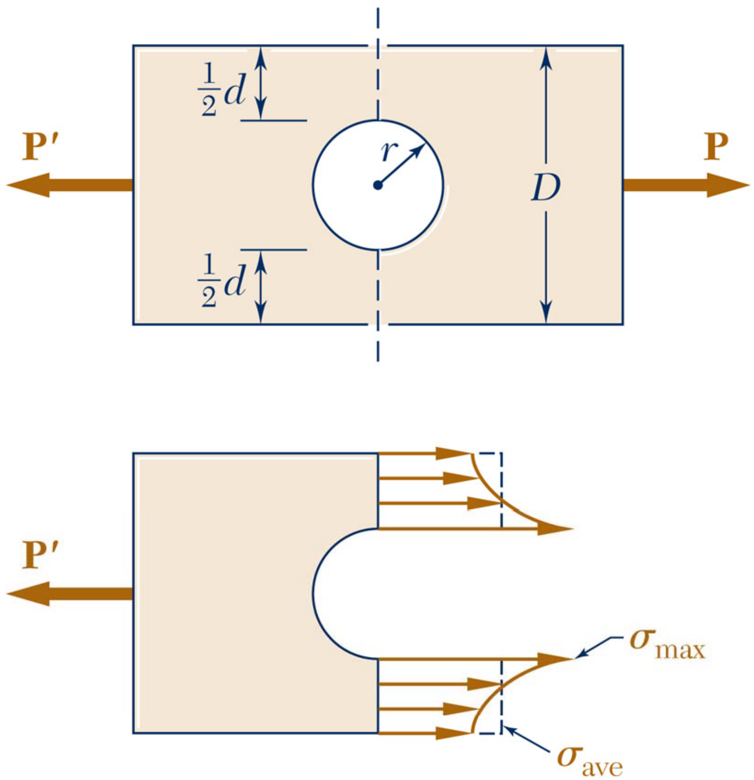

NC STATE achieve! Innovation in Action

NC STATE UNIVERSITY

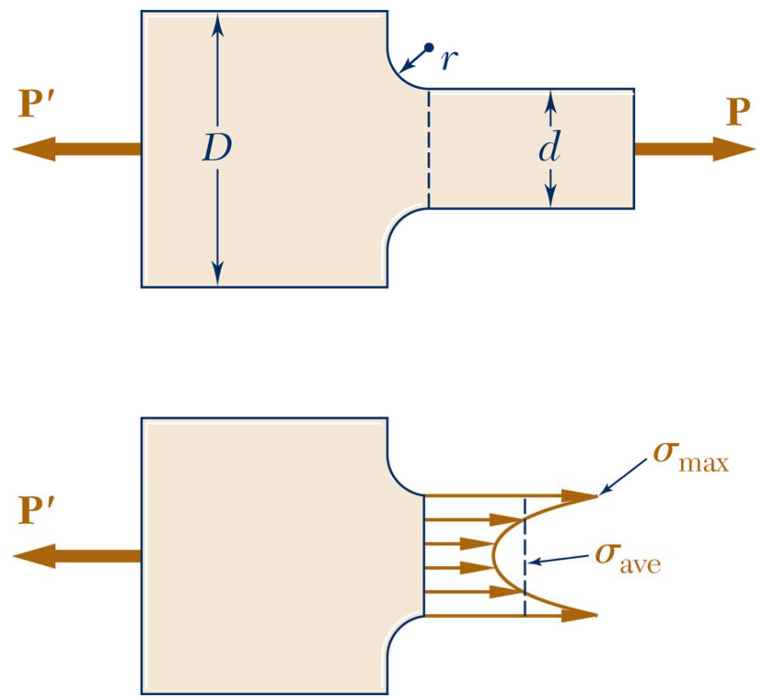
# Stress Concentration

- The stresses near the points of application of concentrated loads can reach values much larger than the average value of the stress in the member.
- Stress concentration factor, $K = \sigma_{max}/\sigma_{ave}$

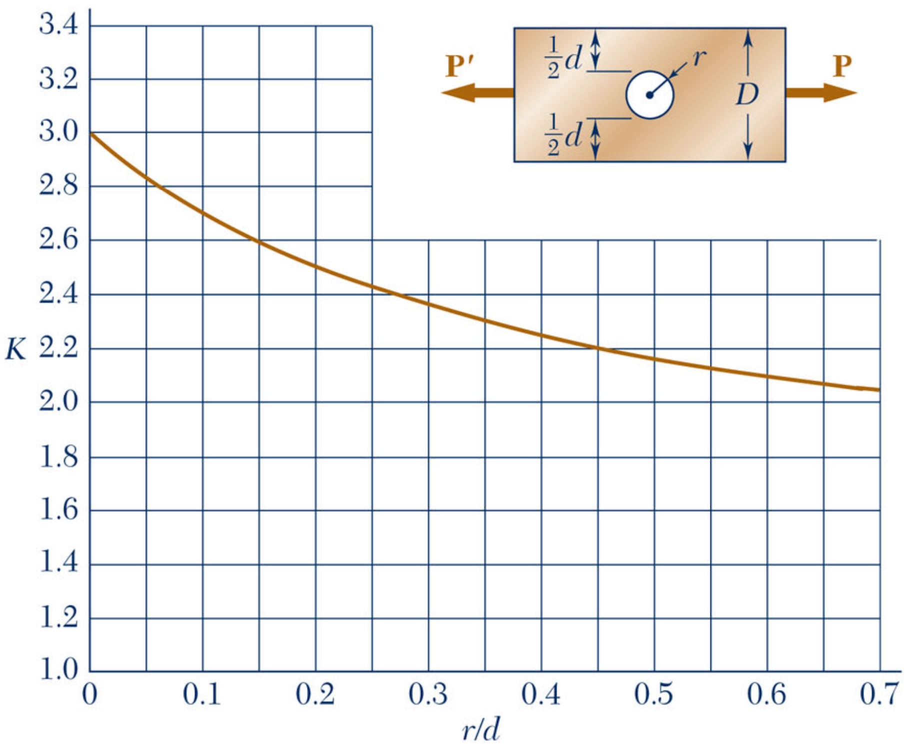
Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display.

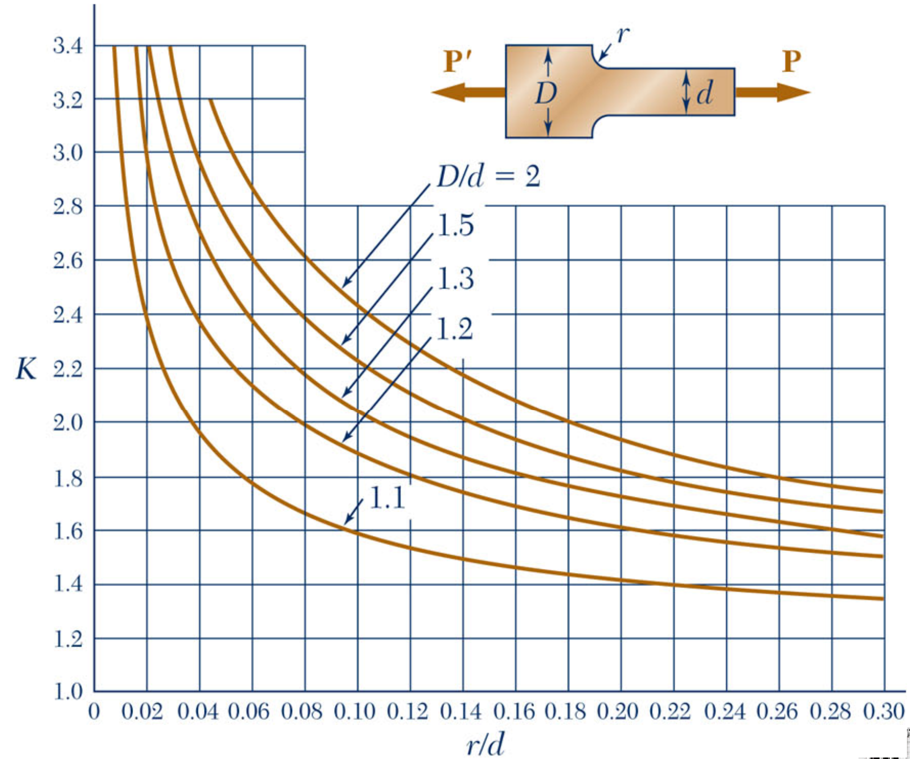



Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display.



Copyright © The McGraw-Hill Companies, Inc. Permission required for reproduction or display.



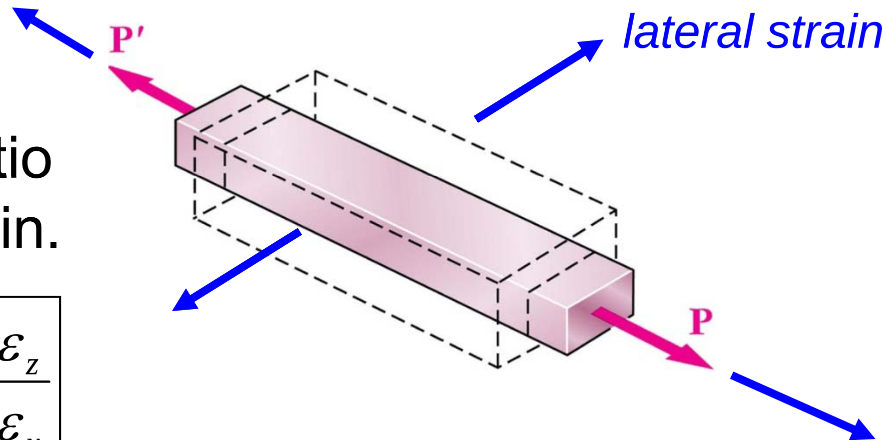



# Example

- Determine the largest axial load P that can safely supported by a flat steel bar consisting of two portions, both 10 mm thick and, respectively, 40 and 60 mm wide, connected by fillets of radius r = 8 mm. assume an allowable normal stress of 165 MPa. (Example 2.12 in Beer's book)

# Poisson's Ratio

- When an axial force is applied to a bar, the bar not only elongates but also shortens in the other two orthogonal directions.




- Poisson's ratio (ν) is the ratio of lateral strain to axial strain.

$$\upsilon = -\frac{lateral\ strain}{axial\ strain} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Minus sign needed to obtain a positive value – all engineering materials have opposite signs for axial and lateral strains

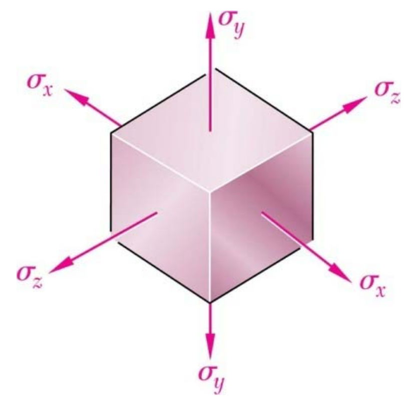
- ν is a material specific property and is dimensionless.

# Generalized Hooke's Law

- Let's generalize Hooke's Law (σ=Eε).
  - Assumptions: linear elastic material, small deformations

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} \qquad \varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$




- So, for the case of a homogenous isotropic bar that is axially loaded along the x-axis ($\sigma_y$=0 and $\sigma_z$=0), we get

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \varepsilon_y = \varepsilon_z = -\frac{\nu\sigma_x}{E}$$

**Even though the stress in the y and z axes are zero, the strain is not!**

# Poisson's Ratio *cont'd*

- What are the limits on ν? We know that ν > 0.
  - Consider a cube with side lengths = 1
  - Apply hydrostatic pressure to the cube

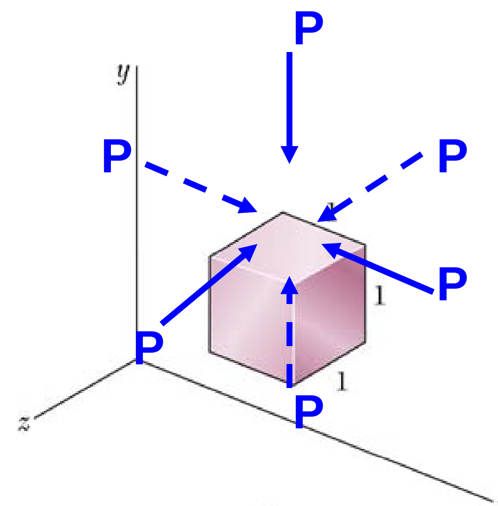
$$\sigma_x = \sigma_y = \sigma_z = -P$$

  - Can write an expression for the change in volume of the cube

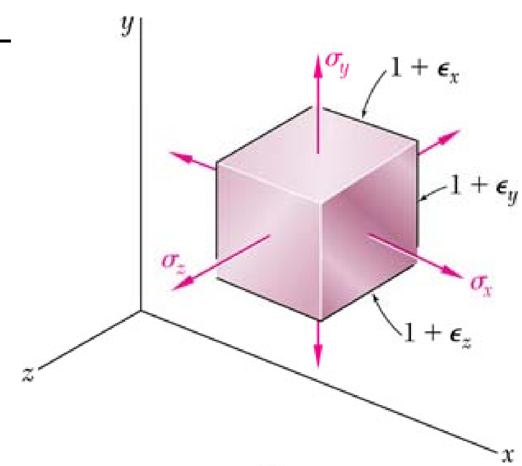
$$\Delta V = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1$$

$$= 1 - 1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \cancelto{0}{\varepsilon_x\varepsilon_y} + \cancelto{0}{\varepsilon_x\varepsilon_z} + \cancelto{0}{\varepsilon_y\varepsilon_z} + \cancelto{0}{\varepsilon_x\varepsilon_y\varepsilon_z}$$

$\varepsilon_x$, $\varepsilon_y$, $\varepsilon_z$ are very small, so we can neglect the terms of order $\varepsilon^2$ or $\varepsilon^3$



(a)



(b)

# Poisson's Ratio *cont'd*

- ΔV simplifies to $\Delta V \cong \varepsilon_x + \varepsilon_y + \varepsilon_z$
- Plug σ=P/A=P into our generalized equations for strain.

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} = -\frac{P}{E} + \frac{\nu P}{E} + \frac{\nu P}{E} = \frac{P}{E}(2\nu - 1)$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} = \frac{\nu P}{E} - \frac{P}{E} + \frac{\nu P}{E} = \frac{P}{E}(2\nu - 1)$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E} = \frac{\nu P}{E} + \frac{\nu P}{E} - \frac{P}{E} = \frac{P}{E}(2\nu - 1)$$

- Plug these values into the expression for ΔV.

$$\Delta V \cong \frac{3P}{E}(2\nu - 1)$$



NC STATE achieve! Innovation in Action

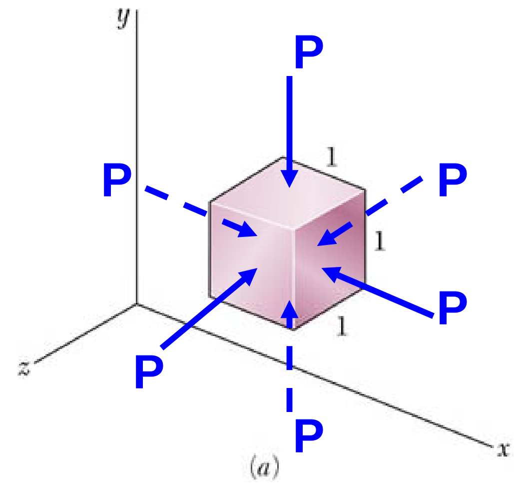
# Poisson's Ratio *cont'd*

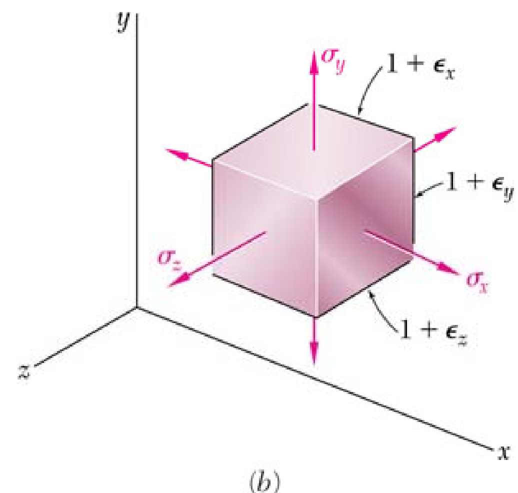
- Since the cube is compressed, we know ΔV must be less than zero.

$$\frac{3P}{E}(2\nu - 1) < 0$$

$$2\nu - 1 < 0$$

$$\boxed{0 < \nu < \frac{1}{2}}$$



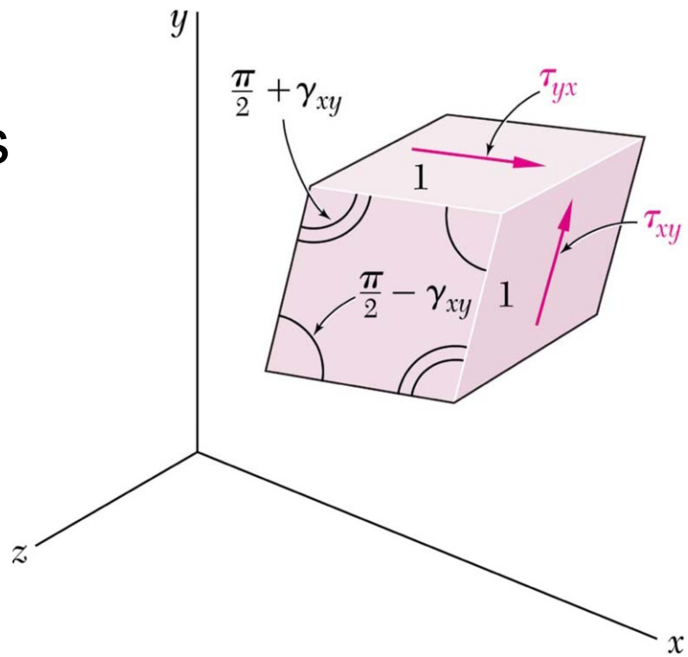



# Shear Strain

- Recall that
  - Normal stresses produce a change in volume of the element
  - Shear stresses produce a change in shape of the element

- Shear strain ($\gamma$) is an angle measured in degrees or radians (dimensionless)

- Sign convention is the same as for shear stress ($\tau$)

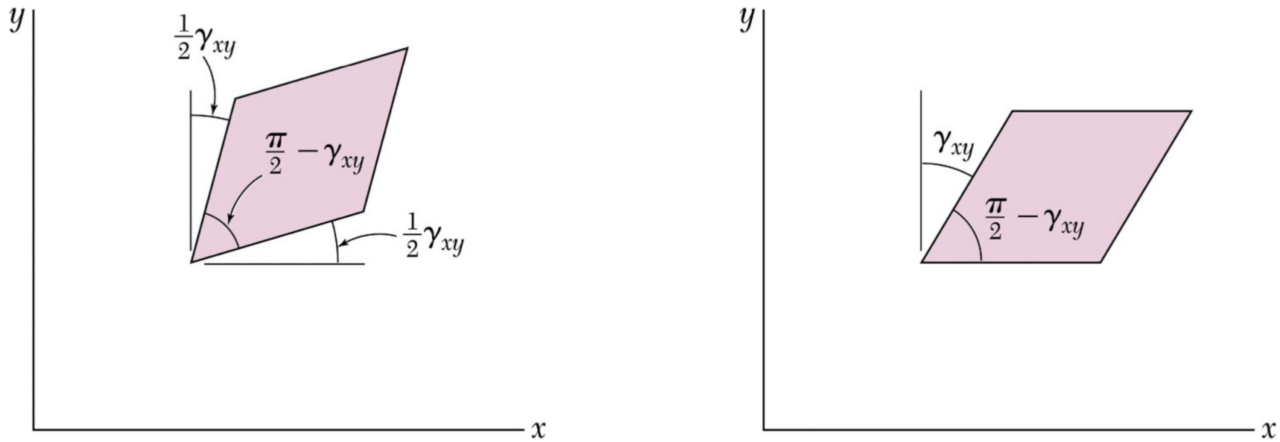





NC STATE achieve! Innovation in Action

# Shear Strain *cont'd*

- There are two equivalent ways to visualize shear strain.




- Hooke's Law for shear stress is defined as

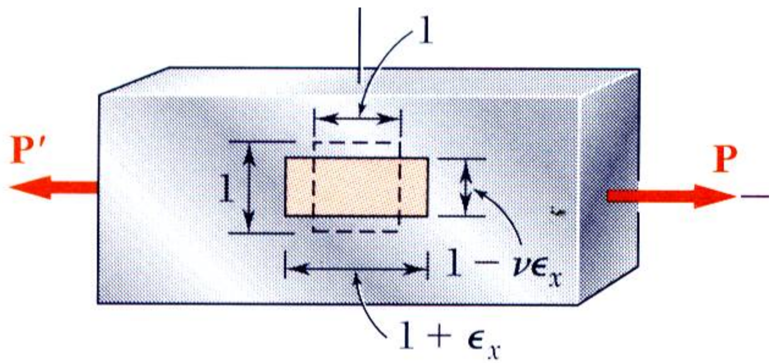
$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{xz} = G\gamma_{xz} \qquad \tau_{yz} = G\gamma_{yz}$$

  - G = shear modulus (or modulus of rigidity)
  - G is a material specific property with the same units as E (psi or Pa).

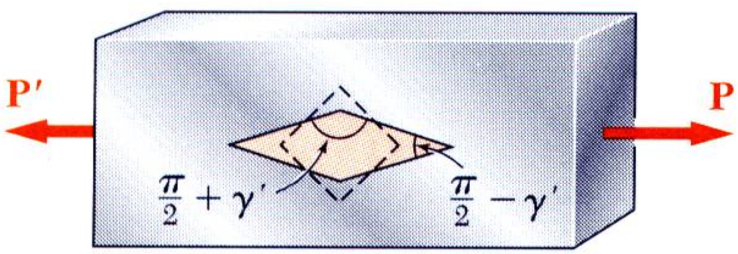


NC STATE achieve! Innovation in Action

# Relation Among *E*, $\nu$, and *G*




(a)

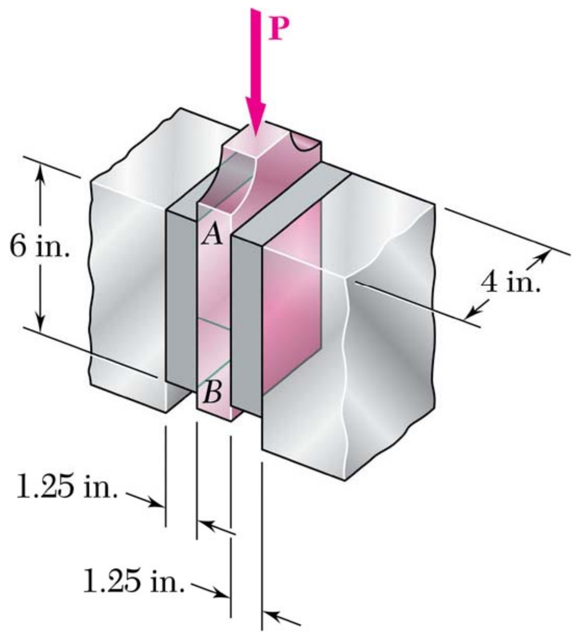



- An axially loaded slender bar will elongate in the axial direction and contract in the transverse directions.
- An initially cubic element oriented as in top figure will deform into a rectangular parallelepiped. The axial load produces a normal strain.
- If the cubic element is oriented as in the bottom figure, it will deform into a rhombus. Axial load also results in a shear strain.
- Components of normal and shear strain are related,

$$\frac{E}{2G} = (1 + \nu)$$

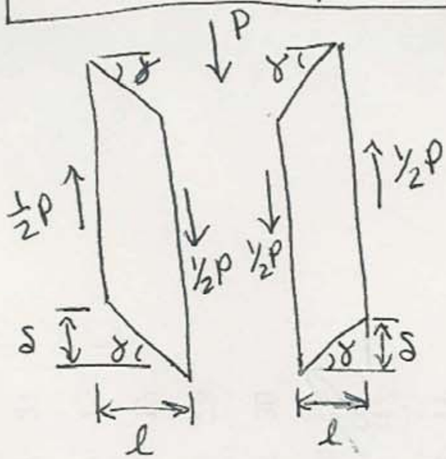
# Example Problem 1

- A vibration isolation unit consists of two blocks of hard rubber bonded to a plate AB and to rigid supports as shown. Knowing that a force of magnitude P = 6 kips causes a deflection of δ=0.0625 in. of plate AB, determine the modulus of rigidity of the rubber used.



# Example Problem 1 Solution

Lect. 6: Example Problem 1




$P = 6\text{ kips} \Rightarrow V = \frac{1}{2}(6\text{ kips}) = 3\text{ kips}$

$\delta = 0.0625\text{ in} \quad l = 1.25\text{ in}$

$A = (6\text{ in})(4\text{ in}) = 24\text{ in}^2$ (Cross section area over which P acts)

$\tau = G\gamma \Rightarrow G = \frac{\tau}{\gamma}$

$$\tau = \frac{V}{A} = \frac{3\text{ kips}}{24\text{ in}^2} = \frac{3000\text{ lb}}{24\text{ in}^2} = 125\text{ psi}$$

$$\gamma = \frac{\delta}{l} = \frac{0.0625\text{ in}}{1.25\text{ in}} = 0.05$$

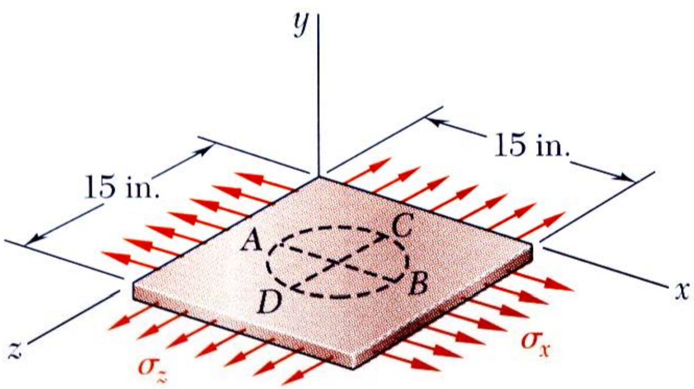
$$G = \frac{125\text{ psi}}{0.05} = 2500\text{ psi}$$

$G = 2.5\text{ ksi}$



NC STATE achieve! Innovation in Action

# Example Problem




A circle of diameter $d = 9$ in. is scribed on an unstressed aluminum plate of thickness $t = 3/4$ in. Forces acting in the plane of the plate later cause normal stresses $\sigma_x = 12$ ksi and $\sigma_z = 20$ ksi.

For $E = 10\text{x}10^6$ psi and $\nu = 1/3$, determine the change in:

a) the length of diameter *AB*,

b) the length of diameter *CD*,

c) the thickness of the plate, and

d) the volume of the plate.

(sample problem 2.5 in Beer's book)



NC STATE achieve! Innovation in Action

# Example Problem 2 Solution

- Apply the generalized Hooke's Law to find the three components of normal strain.

$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= \frac{1}{10\times10^6\,\text{psi}}\left[(12\,\text{ksi}) - 0 - \frac{1}{3}(20\,\text{ksi})\right]$$

$$= +0.533\times10^{-3}\,\text{in./in.}$$

$$\varepsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$= -1.067\times10^{-3}\,\text{in./in.}$$

$$\varepsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= +1.600\times10^{-3}\,\text{in./in.}$$

- Evaluate the deformation components.

$$\delta_{B/A} = \varepsilon_x d = (+0.533\times10^{-3}\,\text{in./in.})(9\,\text{in.})$$

$$\boxed{\delta_{B/A} = +4.8\times10^{-3}\,\text{in.}}$$

$$\delta_{C/D} = \varepsilon_z d = (+1.600\times10^{-3}\,\text{in./in.})(9\,\text{in.})$$

$$\boxed{\delta_{C/D} = +14.4\times10^{-3}\,\text{in.}}$$

$$\delta_t = \varepsilon_y t = (-1.067\times10^{-3}\,\text{in./in.})(0.75\,\text{in.})$$

$$\boxed{\delta_t = -0.800\times10^{-3}\,\text{in.}}$$

- Find the change in volume

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = 1.067\times10^{-3}\,\text{in}^3/\text{in}^3$$

$$\Delta V = eV = 1.067\times10^{-3}(15\times15\times0.75)\text{in}^3$$

$$\boxed{\Delta V = +0.187\,\text{in}^3}$$